

Uhlmann fidelity between two-mode Gaussian states

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We analyze the Uhlmann fidelity of a pair of n -mode Gaussian states of the quantum radiation field. This quantity is shown to be the product of an exponential function depending on the relative average displacement and a factor fully determined by the symplectic spectrum of the covariance matrix of a specific Gaussian state. However, it is difficult to handle our general formula unless the Gaussian states commute or at least one of them is pure. On the contrary, in the simplest cases $n = 1$ and $n = 2$, it leads to explicit analytic formulae. Our main result is a calculable expression of the fidelity of two arbitrary two-mode Gaussian states. This can be applied to build reliable measures of quantum correlations between modes in various branches of quantum physics.

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I. INTRODUCTION.

One of the most important issues in quantum information is the closeness between input and output states of various quantum protocols [1]. When the input state is pure a natural figure of merit of any protocol is the input-output quantum-mechanical transition probability. The situation is more complicated in the mixed-state case. Fortunately, an appropriate description in such a case could be connected with a concept introduced long ago by Uhlmann [2]. Let us consider two arbitrary states, $\hat{\rho}'$ and $\hat{\rho}''$, of a given quantum system. When both states are mixed, Uhlmann asserted that a suitable measure of the likeness between their properties is the maximal quantum-mechanical transition probability between their purifications in an enlarged Hilbert space [2, 3]. This is an extended notion of transition probability between quantum states which is now called fidelity [3] and has the intrinsic expression [2]

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \left[\text{Tr} \left(\sqrt{\sqrt{\hat{\rho}''} \hat{\rho}' \sqrt{\hat{\rho}''}} \right) \right]^2. \quad (1.1)$$

Indeed, when at least one of the quantum states is pure, the fidelity (1.1) reduces to the familiar overlap between the states: $\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \text{Tr}(\hat{\rho}' \hat{\rho}'')$. Important properties listed and discussed in Refs.[2–8] highly recommended fidelity as a measure of distinguishability of quantum states. Especially useful is the property put forward in Ref. [4] that fidelity equals the minimal squared overlap of the probability distributions for the outcomes of any general measurement. Fidelity-based metrics such as the Bures metric [2, 9] and related ones [5–7] have proven to be fruitful in quantum optics and quantum information. For instance, the Bures metric provides insightful distance-type degrees of entanglement [10, 11], nonclassicality [12], and polarization [13–16]. Moreover, in the framework of the theory of parameter estimation, Braunstein and Caves [17] established a distinguishability metric which is proportional to the Bures distance between

neighboring quantum states. Their findings in Ref.[17] successfully apply in recent developments of quantum estimation theory, as shown in the review [18]. Besides being an unavoidable ingredient in quantum information, fidelity is exploited in various areas of physics where the concept of quantum transition probability plays a key role. For instance, in condensed matter physics, fidelity is frequently employed as an indicator of criticality in quantum phase transitions [19, 20].

In spite of playing a key role in quantum mechanics due to its operational meaning, fidelity between mixed states has a rather limited use so far. This happens because, in general, Uhlmann's expression (1.1) is not easy to calculate even on finite-dimensional Hilbert spaces [6, 7]. Within continuous-variable settings, explicit calculations of fidelity were performed for one-mode Gaussian states (GSs) in Refs.[21, 22], for some special two-mode GSs in Refs.[11], or for commuting two-mode states in Refs.[15, 16].

In what follows, we concentrate on the GSs of continuous-variable systems, which are important for both quantum optics and quantum information processing [23]. From the experimental side, they are quite accessible and were effectively obtained with light, Bose-Einstein condensates, trapped ions, and Josephson junctions. On the other hand, the GSs of the quantum radiation field are important resources in several quantum information protocols [24, 25]. In particular, from the theoretical point of view, the two-mode GSs are a perfect test bed for studying entanglement [11] or other kinds of correlations between the two modes [26, 27]. Therefore, an exact analytic formula for the fidelity of an arbitrary pair of two-mode GSs is highly desirable and its derivation constitutes a main goal of the present paper. Let us mention that we have previously derived the fidelity between two-mode symmetric and squeezed thermal states and used it in evaluating a Gaussian degree of entanglement defined with the Bures metric [11]. Nevertheless, an explicit formula of the fidelity between arbitrary two-mode GSs was not yet found. We acknowledge, however,

Ref.[28] as a useful prospect for the multimode Gaussian case.

In order to study the fidelity of two GSs, we find it convenient to introduce the operators:

$$\hat{\mathcal{A}} := \hat{\rho}' \hat{\rho}'', \quad \hat{\mathcal{B}} := \sqrt{\hat{\rho}''} \hat{\rho}' \sqrt{\hat{\rho}''}, \quad \hat{\mathcal{C}} := \sqrt{\hat{\rho}'} \sqrt{\hat{\rho}''}, \quad (1.2)$$

with the properties $\hat{\mathcal{B}} = \hat{\mathcal{C}}^\dagger \hat{\mathcal{C}}$ and $\text{Tr}(\hat{\mathcal{A}}) = \text{Tr}(\hat{\mathcal{B}}) \geq 0$. When this trace is strictly positive, $\text{Tr}(\hat{\mathcal{A}}) > 0$, then a state

$$\hat{\rho}_B := [\text{Tr}(\hat{\mathcal{B}})]^{-1} \hat{\mathcal{B}}, \quad (1.3)$$

exists, so that the fidelity (1.1) reads:

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') = \text{Tr}(\hat{\mathcal{A}}) \left[\text{Tr} \left(\sqrt{\hat{\rho}_B} \right) \right]^2. \quad (1.4)$$

Equation (1.4) is the starting point of our explicit evaluation of the fidelity between two-mode GSs.

From now on, we assume that $\hat{\rho}'$ and $\hat{\rho}''$ are multimode GSs. The plan of the paper is as follows. In Sec. II we recall the phase-space description of the GSs and mention Williamson's theorem, which is at the heart of our method. Some useful properties of the product of two Hilbert-Schmidt Gaussian operators are collected in Sec. III. Based on the composition rules of such a product, we then proceed in Sec. IV to derive the fidelity (1.4) of a pair of multimode GSs. We express the second factor in the formula (1.4) in terms of the symplectic eigenvalues of the covariance matrix (CM) of the GS $\hat{\rho}_B$. We develop our treatment in the multimode framework as far as possible. Then we restrict ourselves to the workable case of two commuting multimode GSs. In Sec. V, the general expression of the fidelity of two GSs enables us to retrieve a previous explicit formula for one-mode states, in agreement with Refs.[21, 22]. Then we reach the main goal of this work: we provide a versatile and workable formula for the fidelity between two-mode GSs. An interesting application of the two-mode formula concludes Sec. V. The final section deals with a discussion of our analytic results and their potential use in studying quantum correlations between modes.

II. PHASE-SPACE DESCRIPTION OF THE GAUSSIAN STATES.

An n -mode Hilbert-Schmidt operator \hat{F} is defined by its Weyl expansion:

$$\hat{F} = \frac{1}{(2\pi)^n} \int d^{2n}u f(u) \hat{D}(-u), \quad (2.1)$$

where u is a vector in the Euclidian space \mathbb{R}^{2n} , the weight function $f(u)$ is square-integrable over \mathbb{R}^{2n} , and $\hat{D}(u)$ is an n -mode Weyl displacement operator:

$$\hat{D}(u) := \bigotimes_{j=1}^n \hat{D}_j((u)_j). \quad (2.2)$$

Let \hat{q}_j and \hat{p}_j be the canonical quadrature operators of the j th field mode, and q_j and p_j their eigenvalues. In Eq. (2.2), $(u)_j$ denotes a vector in \mathbb{R}^2 , whose components are precisely q_j and p_j :

$$(u)_j := \begin{pmatrix} u_{2j-1} \\ u_{2j} \end{pmatrix} = \begin{pmatrix} q_j \\ p_j \end{pmatrix}, \quad (j = 1, 2, \dots, n). \quad (2.3)$$

Further, $\hat{D}_j((u)_j)$ is the corresponding single-mode Weyl displacement operator,

$$\hat{D}_j((u)_j) := \exp[-i(q_j \hat{p}_j - p_j \hat{q}_j)]. \quad (2.4)$$

Recall the standard matrix J of the symplectic form on \mathbb{R}^{2n} , which is block-diagonal and skew-symmetric:

$$J := \bigoplus_{k=1}^n J_k, \quad J_k := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (k = 1, 2, \dots, n). \quad (2.5)$$

Two properties of the Weyl operators (2.2) are useful:

- composition law:

$$\hat{D}(u) \hat{D}(v) = \exp\left(-\frac{i}{2} u^T J v\right) \hat{D}(u+v); \quad (2.6)$$

- orthonormalization relation:

$$\text{Tr} \left[\hat{D}^\dagger(u) \hat{D}(v) \right] = (2\pi)^n \delta^{(2n)}(u-v). \quad (2.7)$$

For instance, the latter yields the scalar product of two Hilbert-Schmidt n -mode operators (2.1),

$$\text{Tr} \left(\hat{F}^\dagger \hat{G} \right) = \frac{1}{(2\pi)^n} \int d^{2n}u f^*(u) g(u), \quad (2.8)$$

as well as the trace-formula for a weight function,

$$f(u) = \text{Tr} \left[\hat{D}(u) \hat{F} \right]. \quad (2.9)$$

Note the value $f(0) = \text{Tr}(\hat{F})$ and that an operator (2.1) is self-adjoint iff $f^*(-u) = f(u)$.

We focus on multimode Hilbert-Schmidt Gaussian operators (HSGOs). By definition, the weight function (2.9) of such an operator is a shifted Gaussian,

$$f(u) = f(0) \exp\left(-\frac{1}{2} u^T \mathcal{F} u - i \xi^T u\right), \quad (2.10)$$

with a complex shift vector $\xi \in \mathbb{C}^{2n}$ and a symmetric matrix $\mathcal{F} \in M_{2n}(\mathbb{C})$ whose real part is positive definite: $\mathcal{F} + \mathcal{F}^* > 0$. This implies that the square matrix \mathcal{F} is invertible, so that the Gaussian weight function (2.10) is integrable [29]:

$$\begin{aligned} & \int d^{2n}u \exp\left(-\frac{1}{2} u^T \mathcal{F} u - i \xi^T u\right) \\ &= (2\pi)^n [\det(\mathcal{F})]^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \xi^T \mathcal{F}^{-1} \xi\right), \\ & \left(\Re e \left\{ [\det(\mathcal{F})]^{\frac{1}{2}} \right\} > 0 \right). \end{aligned} \quad (2.11)$$

A special example of HSGO is the density operator $\hat{\rho}$ of an n -mode GS, which is positive and of unit trace: $\text{Tr}(\hat{\rho}) = 1$. Its weight function (2.9), called the characteristic function (CF) of the state, is fully determined by the first- and second-order moments of all the quadrature operators:

$$\chi(u) = \exp \left[-\frac{1}{2} u^T \mathcal{V} u - i (\langle u \rangle_{\hat{\rho}})^T u \right]. \quad (2.12)$$

In Eq. (2.12), $\mathcal{V} \in M_{2n}(\mathbb{R})$ denotes the symmetric CM of the GS $\hat{\rho}$. It fulfills the concise Robertson-Schrödinger uncertainty relation,

$$\zeta^\dagger \left(\mathcal{V} + \frac{i}{2} J \right) \zeta \geq 0, \quad (\zeta \in \mathbb{C}^{2n}), \quad (2.13)$$

as a necessary and sufficient condition [30, 31]. Therefore, the CM \mathcal{V} is positive definite, $\mathcal{V} > 0$, and, according to Williamson's theorem [32], is congruent via a symplectic matrix $S \in Sp(2n, \mathbb{R})$ to a diagonal matrix:

$$\begin{aligned} \mathcal{V} &= S^T \left(\bigoplus_{j=1}^n \kappa_j I_j \right) S : \quad \det(\mathcal{V}) = \prod_{j=1}^n (\kappa_j)^2 ; \\ \mathcal{V} + \frac{i}{2} J &= S^T \left[\bigoplus_{j=1}^n \left(\kappa_j I_j + \frac{i}{2} J_j \right) \right] S : \\ \det \left(\mathcal{V} + \frac{i}{2} J \right) &= \prod_{j=1}^n \left[(\kappa_j)^2 - \frac{1}{4} \right]. \end{aligned} \quad (2.14)$$

In Eqs. (2.14), the positive numbers κ_j , ($j = 1, 2, \dots, n$), are the symplectic eigenvalues of the CM \mathcal{V} [33], and all the single-mode matrices I_j are equal to the 2×2 identity matrix. As a consequence, the n -mode Robertson-Schrödinger uncertainty relation (2.13) is equivalent to the inequalities $\kappa_j \geq \frac{1}{2}$, ($j = 1, 2, \dots, n$).

III. PRODUCT OF TWO HILBERT-SCHMIDT GAUSSIAN OPERATORS.

We consider the product $\hat{F} = \hat{F}' \hat{F}''$ of two n -mode HSGOs whose weight functions $f'(u)$ and $f''(u)$ are Gaussians, Eq. (2.10). Making use of Eqs. (2.6), (2.7), and (2.11), one finds that the weight function (2.9) of the product \hat{F} has the same form. Although we do not list the corresponding composition rules [34] here, we just stress that the real part of the resulting symmetric matrix $\mathcal{F} \in M_{2n}(\mathbb{C})$ is positive definite in two special cases we are interested in:

1. Both factors \hat{F}' and \hat{F}'' are self-adjoint.
2. The factors are adjoints to one another: $\hat{F}' = (\hat{F}'')^\dagger$.

Hence, in both above-mentioned cases, the product \hat{F} is itself an n -mode HSGO. In the first case, \hat{F} is not self-adjoint, unless the two factors commute. In the second

one, \hat{F} is a positive operator, proportional to the density operator of a GS whose CM is precisely $\mathcal{F} \in M_{2n}(\mathbb{R})$.

We further investigate the product $\hat{\mathcal{A}} := \hat{\rho}' \hat{\rho}''$ of two n -mode GSs. First, making use of Eqs. (2.8)-(2.11), we get its trace:

$$\begin{aligned} \text{Tr}(\hat{\mathcal{A}}) &= [\det(\mathcal{V}' + \mathcal{V}'')]^{-\frac{1}{2}} \\ &\times \exp \left[-\frac{1}{2} (\delta \langle u \rangle)^T (\mathcal{V}' + \mathcal{V}'')^{-1} \delta \langle u \rangle \right] > 0. \end{aligned} \quad (3.1)$$

We have introduced above the relative average n -mode displacement $\delta \langle u \rangle := \langle u \rangle_{\hat{\rho}'} - \langle u \rangle_{\hat{\rho}''}$.

Other properties of the product operator $\hat{\mathcal{A}}$ are found by evaluating its weight function (2.9) via Eq. (2.1) used for both its factors $\hat{\rho}'$ and $\hat{\rho}''$. After finally performing a routine integral of the type (2.11) we are left with the Gaussian weight function $f_A(u)$, Eq. (2.10), which is determined by the following composition rules:

$$\begin{aligned} \mathcal{F}_A &= -\frac{i}{2} J + \left(\mathcal{V}'' + \frac{i}{2} J \right) (\mathcal{V}' + \mathcal{V}'')^{-1} \left(\mathcal{V}' + \frac{i}{2} J \right); \\ \xi_A &= \langle u \rangle_{\hat{\rho}'} - \left(\mathcal{V}' - \frac{i}{2} J \right) (\mathcal{V}' + \mathcal{V}'')^{-1} \delta \langle u \rangle. \end{aligned} \quad (3.2)$$

Similarly, we employ the Hilbert-Schmidt scalar product (2.8) in conjunction with the Gaussian integral (2.11) in order to get the formula

$$\text{Tr}(\hat{\mathcal{A}}^2) = [\text{Tr}(\hat{\mathcal{A}})]^2 2^{-n} [\det(\mathcal{F}_A)]^{-\frac{1}{2}}. \quad (3.3)$$

An essential non-trivial step is that the determinant of the matrix \mathcal{F}_A , Eq. (3.2), always factors as follows:

$$\det(\mathcal{F}_A) = \frac{\det[(J\mathcal{V}')(J\mathcal{V}'') - \frac{1}{4}I]}{\det(\mathcal{V}' + \mathcal{V}'')}. \quad (3.4)$$

The determinant (3.4) easily emerges when arranging the matrix \mathcal{F}_A in a product form. On the other hand, Eq. (3.2) implies the identity

$$\det \left(\mathcal{F}_A + \frac{i}{2} J \right) = \frac{\det(\mathcal{V}' + \frac{i}{2} J) \det(\mathcal{V}'' + \frac{i}{2} J)}{\det(\mathcal{V}' + \mathcal{V}'')}. \quad (3.5)$$

Both the above determinants (3.4) and (3.5) are manifestly symmetric with respect to the GSs $\hat{\rho}'$ and $\hat{\rho}''$.

We exploit the composition rules (3.1) and (3.2) in three special situations. First, when the GSs coincide, $\hat{\rho}' = \hat{\rho}'' =: \hat{\rho}$, then $\mathcal{F}_A = \mathcal{V}_B =: \mathcal{V}_A$. Hence, Eqs. (3.1) and (3.2) read:

$$\text{Tr}(\hat{\rho}^2) = 2^{-n} [\det(\mathcal{V})]^{-\frac{1}{2}} \leq 1; \quad (3.6)$$

$$\mathcal{V}_A = \frac{1}{2} \left(\mathcal{V} - \frac{1}{4} J \mathcal{V}^{-1} J \right), \quad \xi_A = \langle u \rangle_{\hat{\rho}}. \quad (3.7)$$

Second, Eqs. (3.6), (3.7) and (2.14) enable us to write down two conditions, both of them necessary and sufficient in order that an n -mode GS $\hat{\sigma}$ is pure. They are expressed in terms of its CM \mathcal{V} , as follows:

- Scalar condition:

$$\begin{aligned} \text{Tr}(\hat{\sigma}^2) = 1 &\iff \det(\mathcal{V}) = 2^{-2n} \iff \\ &\iff \kappa_j = \frac{1}{2}, \quad (j = 1, 2, \dots, n); \end{aligned} \quad (3.8)$$

- Matrix condition:

$$\mathcal{V}_A = \mathcal{V} \iff (J\mathcal{V})^2 = -\frac{1}{4}I. \quad (3.9)$$

Third, the square root $\sqrt{\hat{\rho}}$ of a given n -mode GS $\hat{\rho}$ with the CF (2.12) is proportional to an equally displaced n -mode GS: its CM $\tilde{\mathcal{V}}$ determines the proportionality factor

$$\text{Tr}(\sqrt{\hat{\rho}}) = \sqrt[4]{\det(2\tilde{\mathcal{V}})}, \quad (3.10)$$

and satisfies Eq. (3.7):

$$\mathcal{V} = \frac{1}{2} \left(\tilde{\mathcal{V}} - \frac{1}{4} J \tilde{\mathcal{V}}^{-1} J \right). \quad (3.11)$$

Accordingly, its symplectic eigenvalues $\tilde{\kappa}_j$ are determined by those of the CM \mathcal{V} , hereafter denoted κ_j :

$$\tilde{\kappa}_j = \kappa_j + \sqrt{(\kappa_j)^2 - \frac{1}{4}}, \quad (j = 1, 2, \dots, n). \quad (3.12)$$

IV. FIDELITY OF TWO MULTIMODE GAUSSIAN STATES.

We successively apply Eqs. (3.10), (2.14), and (3.12) to the GS $\hat{\rho}_B$, Eq. (1.3), for getting a general expression of the fidelity (1.4):

$$\begin{aligned} \mathcal{F}(\hat{\rho}', \hat{\rho}'') &= \text{Tr}(\hat{\rho}' \hat{\rho}'') \\ &\times 2^n \prod_{j=1}^n \left\{ (\kappa_B)_j + \sqrt{[(\kappa_B)_j]^2 - \frac{1}{4}} \right\}. \end{aligned} \quad (4.1)$$

In Eq. (4.1), the set $\{(\kappa_B)_j, (j = 1, 2, \dots, n)\}$ is the symplectic spectrum of the CM \mathcal{V}_B of the n -mode GS $\hat{\rho}_B$. Some expected properties of the fidelity can be read on the above expression. For instance, let us note the inequalities

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') \geq \text{Tr}(\hat{\rho}' \hat{\rho}'') > 0, \quad (4.2)$$

with saturation of the first one when at least one of the GSs $\hat{\rho}'$ and $\hat{\rho}''$ is pure. The second inequality is displayed by Eq. (3.1): the Hilbert-Schmidt scalar product of any pair of n -mode GSs is strictly positive. Another property of the fidelity (4.1) is its symplectic invariance:

$$\mathcal{F}(\hat{U}(S)\hat{\rho}'\hat{U}^\dagger(S), \hat{U}(S)\hat{\rho}''\hat{U}^\dagger(S)) = \mathcal{F}(\hat{\rho}', \hat{\rho}''), \quad (4.3)$$

where $S \in Sp(2n, \mathbb{R})$ and $\hat{U}(S)$ are the unitary operators of the metaplectic representation on the Hilbert space of the n -mode states.

There is a special case when the multimode fidelity (4.1) can be readily evaluated, namely, that of commuting GSs. As a prototype of such states, we mention the n -mode thermal states. We concentrate on a pair of commuting n -mode GSs, $\hat{\rho}'$ and $\hat{\rho}''$, whose CMs \mathcal{V}' and \mathcal{V}'' have the symplectic spectra $\{\kappa'_j\}$ and $\{\kappa''_j\}$, respectively. The commutation relation $[\hat{\rho}', \hat{\rho}''] = \hat{0}$ implies, via the resulting equality $\mathcal{V}_B = \mathcal{F}_A$, that the diagonalizable matrices $J\mathcal{V}'$ and $J\mathcal{V}''$ commute, so that they have a complete system of common eigenvectors. Since the matrix $J\mathcal{V}_B$ has the same eigenvectors, we find the composition law of the symplectic eigenvalues:

$$(\kappa_B)_j = \frac{\kappa'_j \kappa''_j + \frac{1}{4}}{\kappa'_j + \kappa''_j}, \quad (j = 1, 2, \dots, n). \quad (4.4)$$

Accordingly, the fidelity formula (4.1) simplifies to:

$$\begin{aligned} \mathcal{F}(\hat{\rho}', \hat{\rho}'') &= \prod_{j=1}^n \frac{2}{(\kappa'_j + \kappa''_j)^2} \\ &\times \left\{ \kappa'_j \kappa''_j + \frac{1}{4} + \sqrt{\left[(\kappa'_j)^2 - \frac{1}{4}\right] \left[(\kappa''_j)^2 - \frac{1}{4}\right]} \right\}. \end{aligned} \quad (4.5)$$

Equation (4.5) displays two additional properties of the fidelity, namely, its symmetry, $\mathcal{F}(\hat{\rho}'', \hat{\rho}') = \mathcal{F}(\hat{\rho}', \hat{\rho}'')$, and the saturable inequality

$$\mathcal{F}(\hat{\rho}', \hat{\rho}'') \leq 1 : \quad \mathcal{F}(\hat{\rho}', \hat{\rho}'') = 1 \quad \text{iff} \quad \hat{\rho}'' = \hat{\rho}'. \quad (4.6)$$

We come back to the general case of two non-commuting n -mode GSs: writing an explicit formula of their fidelity (4.1) seems to be a hard task. Indeed, evaluation of the CM \mathcal{V}_B of the n -mode GS $\hat{\rho}_B$, Eq. (1.3), requires a three-time application of the composition rule (3.2) and therefore the CM $\mathcal{V}_B \in M_{2n}(\mathbb{R})$ has a complicated structure. However, we succeeded to calculate its symplectic invariants $\det(\mathcal{V}_B)$ and $\det(\mathcal{V}_B + \frac{i}{2}J)$. Indeed, taking advantage once again of Eqs. (2.8) and (2.11), we find a formula which is similar to Eq. (3.3):

$$\text{Tr}(\hat{\mathcal{B}}^2) = [\text{Tr}(\hat{\mathcal{B}})]^2 2^{-n} [\det(\mathcal{V}_B)]^{-\frac{1}{2}}. \quad (4.7)$$

Their comparison gives, via Eq. (1.2), the expression (3.4):

$$\det(\mathcal{V}_B) = \frac{\det[(J\mathcal{V}') (J\mathcal{V}'') - \frac{1}{4}I]}{\det(\mathcal{V}' + \mathcal{V}'')} \geq 2^{-2n}. \quad (4.8)$$

On the other hand, a rather involved calculation makes use of Eqs. (3.2) and (3.11) to yield the determinant

$$\det\left(\mathcal{V}_B + \frac{i}{2}J\right) = \frac{\det(\mathcal{V}' + \frac{i}{2}J) \det(\mathcal{V}'' + \frac{i}{2}J)}{\det(\mathcal{V}' + \mathcal{V}'')}, \quad (4.9)$$

which coincides with the non-negative determinant (3.5).

When at least one of the GSs $\hat{\rho}'$ and $\hat{\rho}''$, say $\hat{\rho}'$, is pure, then insertion of the appropriate conditions (3.8) and

(3.9) into Eq. (4.8) leads to the equality $\det(\mathcal{V}_B) = 2^{-2n}$, showing that the GS $\hat{\rho}_B$ is also pure.

For later convenience, let us introduce the following notations:

$$\begin{aligned}\Delta &:= \det(\mathcal{V}' + \mathcal{V}'') \geq 1; \\ \Gamma &:= 2^{2n} \det \left[(J\mathcal{V}') (J\mathcal{V}'') - \frac{1}{4}I \right] \geq \Delta; \\ \Lambda &:= 2^{2n} \det \left(\mathcal{V}' + \frac{i}{2}J \right) \det \left(\mathcal{V}'' + \frac{i}{2}J \right) \geq 0. \quad (4.10)\end{aligned}$$

The inequality $\Delta \geq 1$ originates in the feature of the positive definite matrix $\mathcal{V} := \frac{1}{2}(\mathcal{V}' + \mathcal{V}'')$ of being the CM of an n -mode GS determined up to its average displacement. Let us remark that our main results in this section are the explicit expressions of the symplectic invariants (4.8) and (4.9) for GSs with any n . However, only for one- and two-mode states these invariants are sufficient to get analytic formulae for the fidelity via Eq. (4.1).

V. EXPLICIT RESULTS

A. One-mode Gaussian states.

For the sake of completeness we begin with the one-mode case. The symplectic spectrum of the CM \mathcal{V}_B consists of a single eigenvalue,

$$\kappa_B = \frac{1}{2} \sqrt{\frac{\Lambda}{\Delta} + 1}.$$

According to Eqs. (3.2) and (4.10), the fidelity (4.1) reads:

$$\begin{aligned}\mathcal{F}(\hat{\rho}', \hat{\rho}'') &= \exp \left[-\frac{1}{2} (\delta\langle u \rangle)^T (\mathcal{V}' + \mathcal{V}'')^{-1} \delta\langle u \rangle \right] \\ &\times \left(\sqrt{\Delta + \Lambda} - \sqrt{\Lambda} \right)^{-1}. \quad (5.1)\end{aligned}$$

The symmetric formula (5.1) was derived and analyzed gradually long ago in Refs. [22]. In addition, by using the structure of the one-mode GSs, which are displaced squeezed thermal states [35], we succeeded to prove explicitly in the present work the property (4.6) of their fidelity (5.1).

B. Two-mode Gaussian states.

The CM \mathcal{V}_B has two symplectic eigenvalues, $(\kappa_B)_1$ and $(\kappa_B)_2$. Their squares are the roots of a monic quadratic binomial whose coefficients are expressed in terms of the determinants (4.10). A straightforward calculation exploiting Eqs. (3.2) and (4.1) leads to the symmetric for-

mula

$$\begin{aligned}\mathcal{F}(\hat{\rho}', \hat{\rho}'') &= \exp \left[-\frac{1}{2} (\delta\langle u \rangle)^T (\mathcal{V}' + \mathcal{V}'')^{-1} \delta\langle u \rangle \right] \\ &\times \left[\left(\sqrt{\Gamma} + \sqrt{\Lambda} \right) - \sqrt{\left(\sqrt{\Gamma} + \sqrt{\Lambda} \right)^2 - \Delta} \right]^{-1}. \quad (5.2)\end{aligned}$$

Previous results reported in Refs.[11] for special states (squeezed thermal and symmetric) are seen to be particular cases of Eq. (5.2).

The determinants (4.10) of 4×4 matrices occurring in Eq. (5.2) are easily computable. As a salient example, we evaluate them for a pair of undisplaced two-mode GSs whose CMs \mathcal{V}' and \mathcal{V}'' are both in an unscaled standard form. This means that they are partitioned into 2×2 diagonal submatrices of the type

$$\mathcal{V} = \begin{pmatrix} b_1 I & C \\ C & b_2 I \end{pmatrix} : \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}. \quad (5.3)$$

Here $b_1 \geq \frac{1}{2}$, $b_2 \geq \frac{1}{2}$, $c \geq |d|$. A matrix (5.3) has two symplectic invariants:

$$\begin{aligned}\det(\mathcal{V}) &= (b_1 b_2 - c^2) (b_1 b_2 - d^2) > 0, \\ \det \left(\mathcal{V} + \frac{i}{2}J \right) &= \det(\mathcal{V}) - \frac{1}{4} (b_1^2 + b_2^2 + 2cd) + \frac{1}{16} \geq 0. \quad (5.4)\end{aligned}$$

We denote b'_1, b'_2, c', d' and b''_1, b''_2, c'', d'' the standard-form parameters of the CMs \mathcal{V}' and \mathcal{V}'' , respectively. Then, Eqs. (4.10), (5.3), and (5.4) allow us to factor both determinants Δ and Γ and to visualize the symplectic invariant Λ :

$$\begin{aligned}\Delta &= \left[(b'_1 + b''_1) (b'_2 + b''_2) - (c' + c'')^2 \right] \\ &\times \left[(b'_1 + b''_1) (b'_2 + b''_2) - (d' + d'')^2 \right]; \\ \Gamma &= 16 \left\{ \left[b'_1 b'_2 - (d')^2 \right] \left[b''_1 b''_2 - (d'')^2 \right] \right. \\ &\quad \left. + \frac{1}{4} (b'_1 b''_1 + b'_2 b''_2 + 2d' c'') + \frac{1}{16} \right\} \\ &\times \left\{ \left[b'_1 b'_2 - (c')^2 \right] \left[b''_1 b''_2 - (d'')^2 \right] \right. \\ &\quad \left. + \frac{1}{4} (b'_1 b''_1 + b'_2 b''_2 + 2c' d'') + \frac{1}{16} \right\}; \\ \Lambda &= 16 \left\{ \det(\mathcal{V}') - \frac{1}{4} [(b'_1)^2 + (b'_2)^2 + 2c' d'] + \frac{1}{16} \right\} \\ &\times \left\{ \det(\mathcal{V}'') - \frac{1}{4} [(b''_1)^2 + (b''_2)^2 + 2c'' d''] + \frac{1}{16} \right\}. \quad (5.5)\end{aligned}$$

We are left to substitute the formulae (5.5) into Eq. (5.2) written for undisplaced two-mode GSs.

VI. CONCLUSIONS

To sum up, in this work we have tackled a long-standing problem in quantum information with contin-

uous variables: the Uhlmann fidelity of two multimode GSs. Our simple phase-space approach led to the important formulae (3.3)–(3.5) and (4.7)–(4.9) which are valid for any number of modes and could be useful for future research. However, we stress that an analytic result for n -mode GSs seems hard to be found when $n \geq 3$: indeed, one then needs to solve a higher-degree characteristic equation that cannot be easily obtained.

While in the single-mode case an explicit formula was found many years ago, we derived here a computable analytic expression in the two-mode case, Eq. (5.2). We expect that this formula will open a productive research in quantifying bipartite quantum correlations (entanglement and discord) in the Gaussian scenario. An example at hand is the already performed evaluation of a de-

gree of entanglement for some important classes of GSs (squeezed thermal states, symmetric states) in Refs.[11]. The analysis of an arbitrary two-mode GS requires our present result. Use of fidelity in evaluating the Gaussian geometric discord defined with the Bures metric is currently one of our main interests. Note also that fidelity in mixed-state estimation or reconstruction could be a potential important tool. It appears that applications of the fidelity to the emerging field of quantum information and to other interesting areas of quantum physics are far from being exhausted.

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